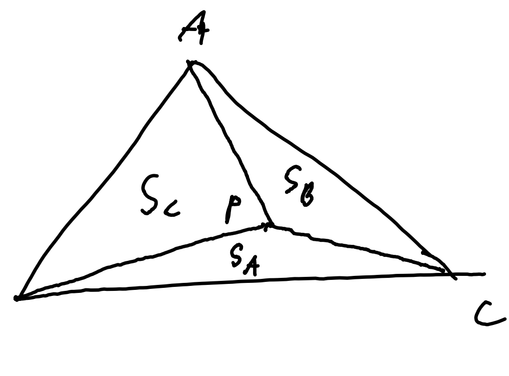
Coordonate baricentrice

Teoremă

Fie ΔABC şi P un punct în interiorul acestuia.

Atunci există şi sunt unice numerele pozitive p_A, p_B, p_C a.î.

$$\begin{cases} p_A + p_B + p_C = 1 \\ (\forall) X \in \mathcal{P}, \quad p_A \cdot \overrightarrow{XA} + p_B \cdot \overrightarrow{XB} + p_C \cdot \overrightarrow{XC} = \vec{0} \end{cases}$$



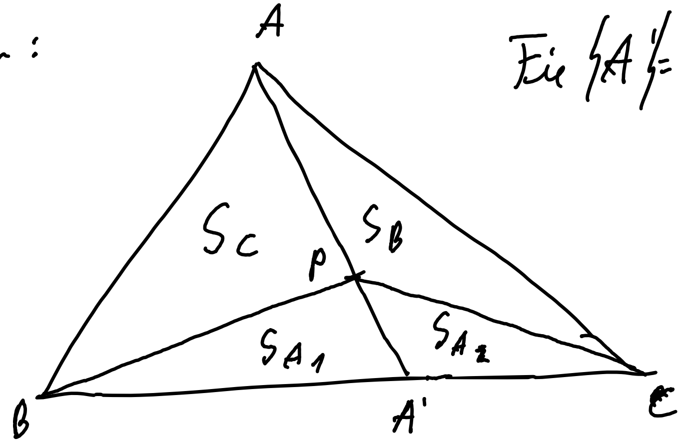
Avem următoarele expresii pentru p_A, p_B, p_C

$$p_A = \frac{S_A}{S} \qquad p_C = \frac{S_C}{S} \qquad p_B = \frac{S_B}{S}$$

unde $\begin{cases} S = A_{\Delta ABC} \\ S_A = A_{\Delta PBC} \\ \text{etc.} \end{cases}$

Dem:

Fie $\{A'\} = AP \cap BC$



Fie $k = \dfrac{AP}{PA'} = \dfrac{S_C}{S_{A_1}}$

Pentru orice $X \in \mathcal{P}$,

$$\overrightarrow{XP} = \frac{k}{k+1} \cdot \overrightarrow{XA'} + \frac{1}{k+1} \cdot \overrightarrow{XA} \quad (1)$$

Fie $l = \dfrac{BA'}{A'C} = \dfrac{S_{A_1}}{S_{A_2}} = \dfrac{S_C + S_{A_1}}{S_B + S_{A_2}}$

$$\overrightarrow{XA'} = \frac{l}{l+1} \cdot \overrightarrow{XC} + \frac{1}{l+1} \cdot \overrightarrow{XB} \quad (2)$$

Punând cap la cap (1) și (2), obținem:

$$\overrightarrow{XP} = \frac{k}{k+1}\left(\frac{l}{l+1} \cdot \overrightarrow{XC} + \frac{1}{l+1} \cdot \overrightarrow{XB} \right) +$$

$$+ \frac{1}{k+1} \cdot \overrightarrow{XA}$$

Calculăm: $\dfrac{k}{k+1} \cdot \dfrac{l}{l+1} =$

$$= \frac{\dfrac{S_C}{S_{A_1}}}{1 + \dfrac{S_C}{S_{A_1}}} \cdot \frac{\dfrac{S_C + S_{A_1}}{S_B + S_{A_2}}}{\dfrac{S_C + S_{A_1}}{S_B + S_{A_2}} + 1}$$

$$= \frac{S_C}{\cancel{S_{A_1}}} \cdot \frac{\cancel{S_{A_1}}}{\cancel{S_{A_1} + S_C}} \cdot \frac{\cancel{S_C + S_{A_1}}}{\cancel{S_B + S_{A_2}}} \cdot \frac{\cancel{S_B + S_{A_2}}}{S_C + S_B + S_A}$$

$$= \frac{S_C}{S}$$

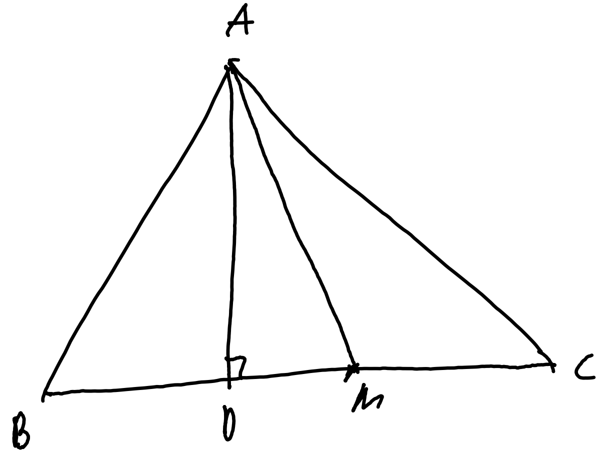
Analog calculele pentru ceilalți coeficienți. $\square$

Am folosit următorul rezultat:

ΔABC

$M \in BC$

Atunci $\dfrac{BM}{MC} = \dfrac{A_{\Delta AMB}}{A_{\Delta AMC}}$



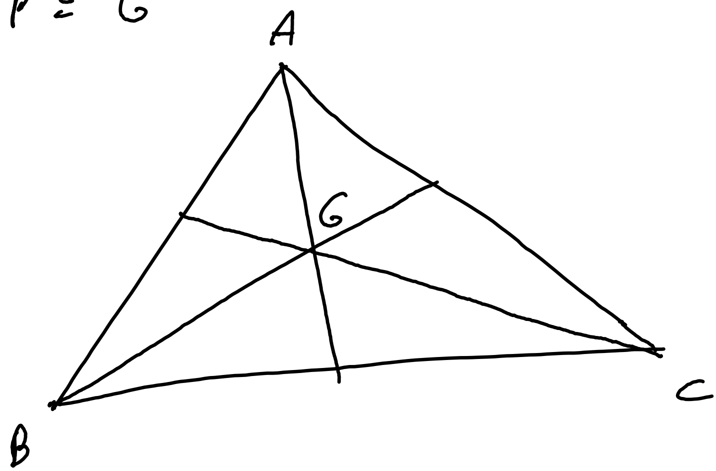
Dem: Fie $AD \perp BC$, $D \in BC$

$$\frac{A_{\Delta AMB}}{A_{\Delta AMC}} = \frac{\frac{BM \cdot AD}{2}}{\frac{MC \cdot AD}{2}} = \frac{BM}{MC}$$ $\square$

Consecințe:

Aplicăm rezultatul pt. punctele importante.

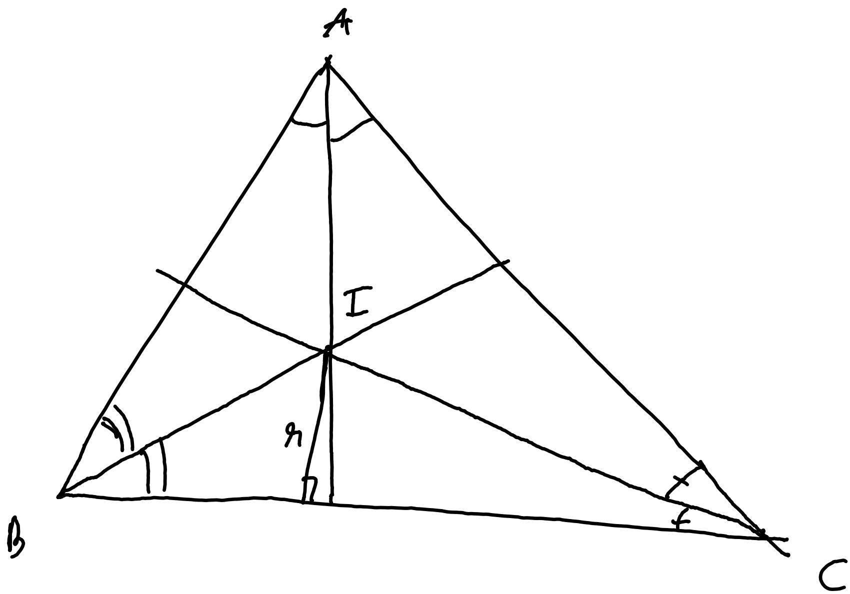
1) $P = G$



$$S_A = S_B = S_C = \frac{S}{3}$$

Deci $\overrightarrow{XG} = \frac{1}{3}\left(\overrightarrow{XA} + \overrightarrow{XB} + \overrightarrow{XC}\right)$

2) $P = I$

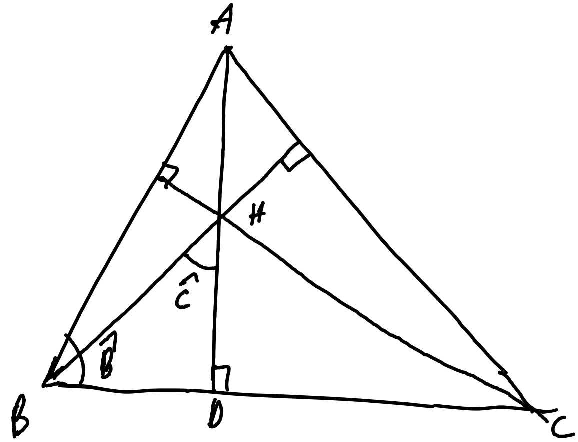


$$\frac{S_A}{S} = \frac{\frac{r \cdot BC}{2}}{S} = \frac{\frac{r \cdot a}{2}}{r \cdot \frac{a+b+c}{2}}$$

$$= \frac{a}{a+b+c}$$

Deci $\overrightarrow{XI} = \frac{1}{a+b+c}\left(a\,\overrightarrow{XA} + b\,\overrightarrow{XB} + c \cdot \overrightarrow{XC}\right)$

3) $P = H$



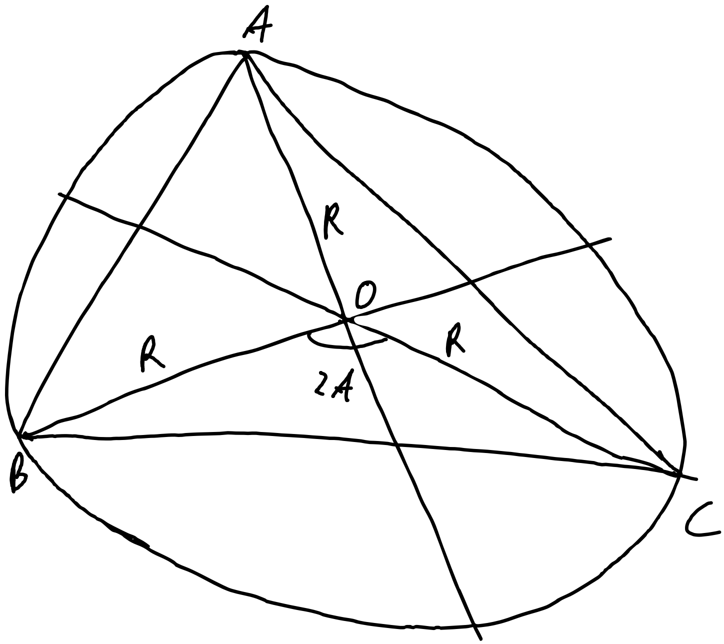
$$\frac{S_A}{S} = \frac{HD}{AD} = \frac{\frac{BD}{tg\,\hat{C}}}{BD \cdot tg\,\hat{B}}$$

$$= \frac{1}{tg\,\hat{C} \cdot tg\,\hat{B}}$$

Deci $\overrightarrow{XH} = \frac{1}{tg\,\hat{A} \cdot tg\,\hat{B} \cdot tg\,\hat{C}} \cdot$

$$\cdot \left(tg\,\hat{A} \cdot \overrightarrow{XA} + tg\,\hat{B} \cdot \overrightarrow{XB} + tg\,\hat{C} \cdot \overrightarrow{XC} \right)$$

4) $X = O$



$$\frac{S_A}{S} = \frac{\frac{R^2 \cdot \sin 2\hat{A}}{2}}{\frac{R^2}{2} \cdot (\sin 2\hat{A} + \sin 2\hat{B} + \sin 2\hat{C})}$$

Deci $\overrightarrow{XO} = \frac{1}{\sin 2\hat{A} + \sin 2\hat{B} + \sin 2\hat{C}} \cdot$

$\cdot \left(\sin 2\hat{A} \cdot \overrightarrow{XA} + \sin 2\hat{B} \cdot \overrightarrow{XB} + \sin 2\hat{C} \cdot \overrightarrow{XC} \right)$

Relația lui Euler

$$OI^2 = R^2 - 2Rr$$

De mai sus

$$\vec{OI} = \frac{1}{a+b+c} \cdot \left(a\vec{OA} + b \cdot \vec{OB} + \vec{OC}\right)$$

Ridicăm la pătrat ca produs scalar:

$$OI^2 = \frac{1}{(a+b+c)^2} \cdot \left(a^2R^2 + b^2R^2 + c^2R^2 +\right.$$

$$+ 2ab \cdot \vec{OA} \cdot \vec{OB} + 2bc \vec{OB} \cdot \vec{OC} +$$

$$\left.+ 2ac \cdot \vec{OA} \cdot \vec{OC}\right)$$

$$\vec{OA} \cdot \vec{OB} = \frac{\vec{OA}^2 + \vec{OB}^2 - (\vec{OA} - \vec{OB})^2}{2}$$

$$= \frac{2R^2 - c^2}{2} = R^2 - \frac{c^2}{2}$$

Rezultă că

$$OI^2 = \frac{1}{(a+b+c)^2} \cdot \Big(R^2 (a^2+b^2+c^2+2ab+2ac+2bc) -$$

$$- abc(a+b+c) \Big)$$

$$= R^2 - \frac{abc}{a+b+c}$$

$$S = \frac{abc}{4R} = r \cdot \frac{a+b+c}{2},$$

deci $\frac{abc}{a+b+c} = 2Rr$

În concluzie, $OI^2 = R^2 - 2Rr$

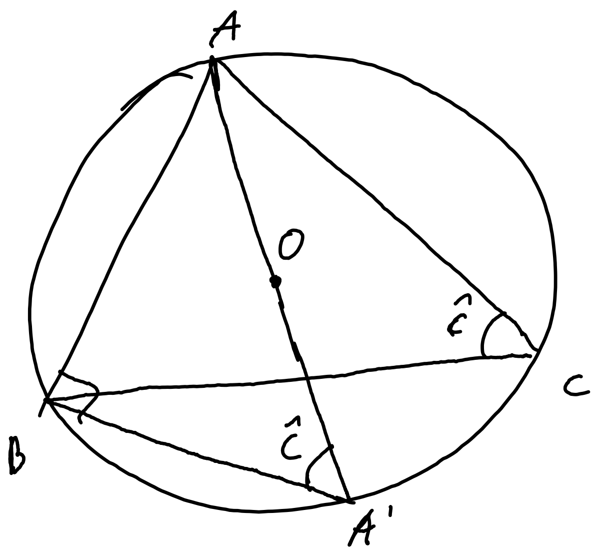
De ce $S = \dfrac{abc}{4R}$?

$$S = \frac{bc \cdot \sin A}{2},$$

deci rămâne să demonstrăm

<u>Teorema sinusurilor</u>

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

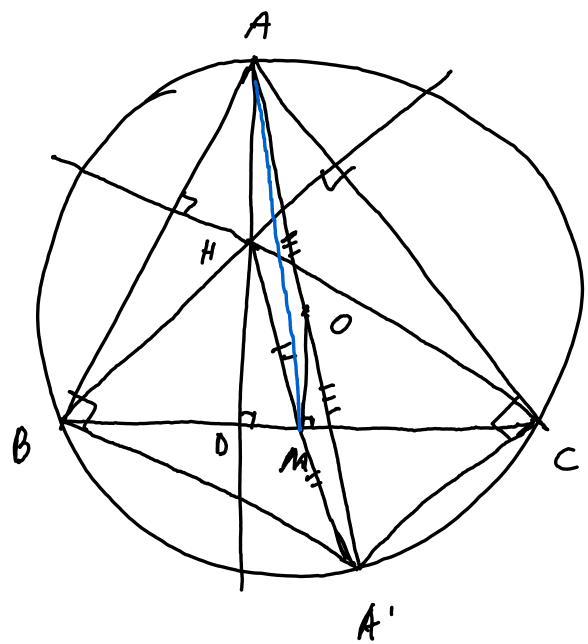


Fie $\{A', A\} = AO \cap \mathcal{C}(A, B, C)$

$$\sin \hat{C} = \frac{AB}{AA'} = \frac{c}{2R}$$

$\square$

Dreapta lui Euler și Relația lui Sylvester



$\left.\begin{array}{l} HC \perp AB \\ A'B \perp AB \end{array}\right| \Rightarrow \left.\begin{array}{l} HC \parallel BA' \\ \text{Analog } HB \parallel A'C \end{array}\right| \Rightarrow$

$\Rightarrow HBA'C$ paralelogram $\Big/ \Rightarrow$

Fie M - mij $[BC]$

$\Rightarrow M$ - mij $[HA']$

$[AM]$, $[HO]$ - mediane în $\Delta A'AH \Rightarrow$

$\Rightarrow$ Dacă $\{G'\} = AM \cap HO$ atunci

$$\frac{AG'}{AM} = \frac{2}{3},$$

deci $G = G'$

În concluzie, $G \in [HO]$ și

$$\frac{OG}{OH} = \frac{1}{3}$$

Vectorial, obținem

$$\overrightarrow{OH} = 3\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

(Relația lui Sylvester)

Probleme:

1) Fie ΔABC neisoscel, M în interiorul ΔABC

G centrul de greutate

I centrul cercului înscris.

Atunci M, I, G coliniare dacă și numai dacă

$$(b-c)A_{\Delta MBC} + (c-a)A_{\Delta MAC} + (a-b)A_{\Delta MAB} = 0$$

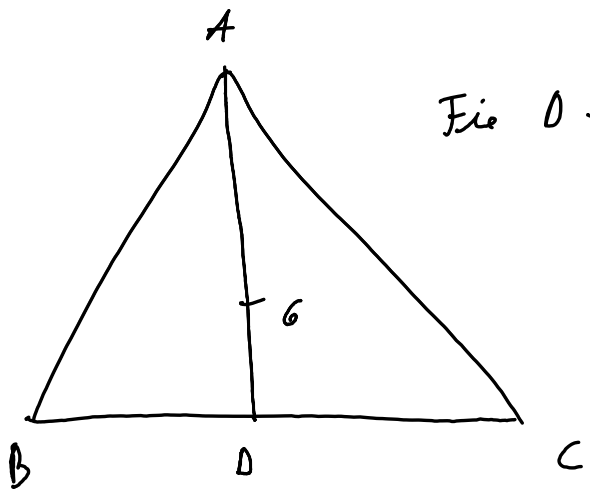
Notăm $S_A = A_{\Delta MBC}$

$S_B = A_{\Delta MAC}$

$S_C = A_{\Delta MAB}$

Avem $\overrightarrow{GM} = \frac{S_A}{S} \cdot \overrightarrow{GA} + \frac{S_B}{S} \cdot \overrightarrow{GB} + \frac{S_C}{S} \cdot \overrightarrow{GC}$

$$\overrightarrow{GI} = \frac{1}{a+b+c}\left(a \cdot \overrightarrow{GA} + b \cdot \overrightarrow{GB} + c \cdot \overrightarrow{GC}\right)$$



Fie D - mij [BC]

$$\vec{GA} = -\frac{2}{3}\cdot\vec{AM} = -\frac{2}{3}\left(\frac{\vec{AB}+\vec{AC}}{2}\right)$$

$$= -\frac{\vec{AB}+\vec{AC}}{3}$$

$$\vec{GB} = -\frac{\vec{BA}+\vec{BC}}{3}$$

$$= -\frac{2\vec{BA}+\vec{AC}}{3}$$

$$= \frac{2\vec{AB}-\vec{AC}}{3}$$

$$\vec{GC} = -\frac{\vec{CA}+\vec{CB}}{3} = -\frac{2\vec{CA}+\vec{AB}}{2}$$

$$= \frac{2\vec{AC}-\vec{AB}}{3}$$

Prin urmare,

$$\vec{GM} = \left(-\frac{1}{3}\cdot\frac{S_A}{S} + \frac{2}{3}\cdot\frac{S_B}{S} - \frac{1}{3}\cdot\frac{S_C}{S}\right)\vec{AB} +$$

$$+\left(-\frac{1}{3}\cdot\frac{S_A}{S} - \frac{1}{3}\frac{S_B}{S} + \frac{2}{3}\frac{S_C}{S}\right)\cdot\vec{AC}$$

$$\vec{GI} = \frac{1}{a+b+c}\cdot\left(-\frac{1}{3}a + \frac{2}{3}b - \frac{1}{3}c\right)\cdot\vec{AB} +$$

$$+\left(-\frac{1}{3}a - \frac{1}{3}b + \frac{2}{3}c\right)\cdot\vec{AC}$$

Prin urmare, $\vec{GM} = \lambda\cdot\vec{GI} \Leftrightarrow$

$$\Leftrightarrow \left\{\begin{array}{l} \dfrac{(-S_A + 2S_B - S_C)}{(-a+2b-c)} = \dfrac{\lambda(-S_A - S_B + 2S_C)}{\lambda(-a-b+2c)} \end{array}\right. \Leftrightarrow$$

$$\overset{(*)}{\Longleftrightarrow} \left(-S_A + 2S_B - S_C\right)\cdot\left(-a - b + 2c\right) =$$

$$= \left(-S_A - S_B + 2S_C\right)\cdot\left(-a + 2b - c\right) \Rightarrow$$

$$\Longleftrightarrow aS_A - 2aS_B + aS_C + bS_A - 2bS_B + bS_C$$

$$- 2cS_A + 4cS_B - 2cS_C =$$

$$= aS_A + aS_B - 2aS_C - 2bS_A - 2bS_B + 4bS_C$$

$$+ cS_A + cS_B - 2cS_C \Longleftrightarrow$$

$$\Longleftrightarrow -3aS_B + 3aS_C + 3bS_A - 3bS_C - 3cS_A$$

$$-3cS_C + 3cS_B = 0 \Longleftrightarrow$$

$$\Longleftrightarrow S_B(c-a) + S_A(b-c) + S_C(a-b) = 0$$

(*) Implicația inversă.

Dacă

$$(-S_A + 2S_B - S_C)\cdot(-a-b+2c) =$$

$$= (-S_A - S_B + 2S_C)\cdot(-a+2b-c)$$

și ambii $(-a-b+2c)$, $(-a+2b-c)$ sunt nenuli $\Rightarrow$

$$\Rightarrow \lambda = \frac{-S_A+2S_B-S_C}{-a+2b-c} = \frac{-S_A-S_B+2S_C}{-a-b+2c}$$

Dacă $(-a-b+2c)=0$ și $(-a+2b-c)\neq 0$

atunci $(-S_A - S_B + 2S_C) = 0$

și iar $\lambda = \dfrac{-S_A+2S_B-c}{c}$

Analog $(-a-b+2c)\neq 0$ și $(-a+2b-c)=0$

$-a-b+2c=-a+2b-c=0$

este imposibil, căci triunghiul nu e isoscel $\square$